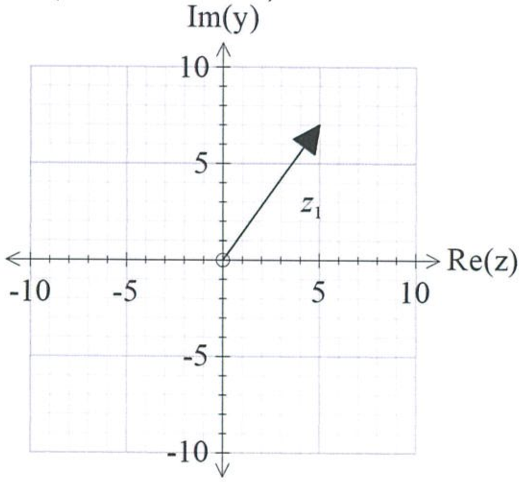
PERTH MODERN SCHOOL Exceptional schooling. Exceptional students. Independent Public School	Year 12 Specialist TEST 2 Monday 11 March 2019 TIME: 45 minutes working Classpads allowed One page of notes 45 marks 7 Questions

Name: Marking Key

Teacher: ____________________

Note: All part questions worth more than 2 marks require working to obtain full marks.

Q1 (2 & 3 = 5 marks)



From the diagram, z_1 is a solution to $z^4 = k$ for complex k.

i) Determine k.

$z^4 = (5+7i)^4 = k$

$= -4324 - 3360i$

✓ z_1 stated
✓ K value

ii) Determine the other three roots and express in the form $a+bi$.

$z_2 = (5+7i)i = -7+5i$

$z_3 = -5-7i$

$z_4 = (-5-7i)i = 7-5i$

✓ shows that each differ by $\times i$

✓ states two correct roots (other than z_1)

✓ states all correct roots

Q2 (2, 3 & 1 = 6 marks)

Let $f(x)=\sqrt{2x-1}$ and $g(x)=\dfrac{1}{x+5}$.

a) State the natural domain and range of $g(x)$.

$d_g: x \neq -5$ ✓

$r_g: y \neq 0$ ✓

b) Does $f \circ g(x)$ exist over the natural domain of g? If it does not, determine the largest possible domain for the composite to exist.

$d_f: x \geq \frac{1}{2}$

$r_g: y \neq 0$

$r_g \not\subseteq d_f \therefore f \circ g$ does not exist

$-5 < x \leq -3$

✓ Explains using GIVEN r_g & d_f

✓ states not exist

✓ new domain $-5 < x \leq -3$

c) Determine $f \circ f^{-1}(x)$

x ✓

Q3 (2, 3 & 2 = 7 marks)

Given that $f(x)=2x^2-12x+19$, $x \leq 3$, determine the following.

a) $f^{-1}(x)$ and its domain.

$x = 2y^2 - 12y + 19$

$0 = 2y^2 - 12y + 19 - x$

$y = \dfrac{12 \pm \sqrt{144 - 4(2)(19-x)}}{4} = \dfrac{12 \pm 2\sqrt{36 - 38 + 2x}}{4}$

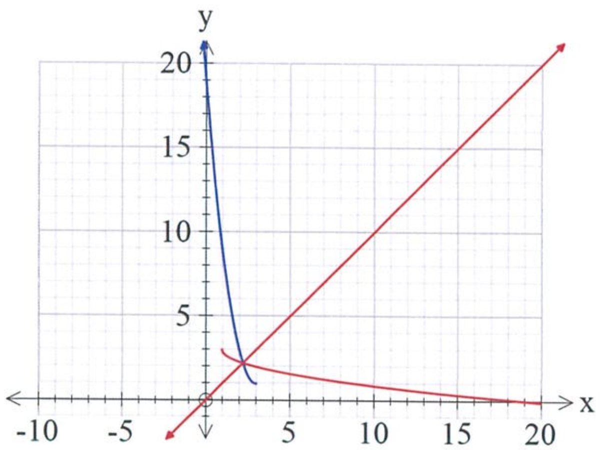
$f^{-1}(x) = 3 - 0.5\sqrt{2x-2}$

$x \geq 1$

✓ rule with negative (No need to simplify)

✓ domain.

b) Sketch on the axes below, $f(x)$ & $f^{-1}(x)$



✓ appears to be reflected in $y=x$

✓ x intercept

✓ overlap between $1 \leq x \leq 4$

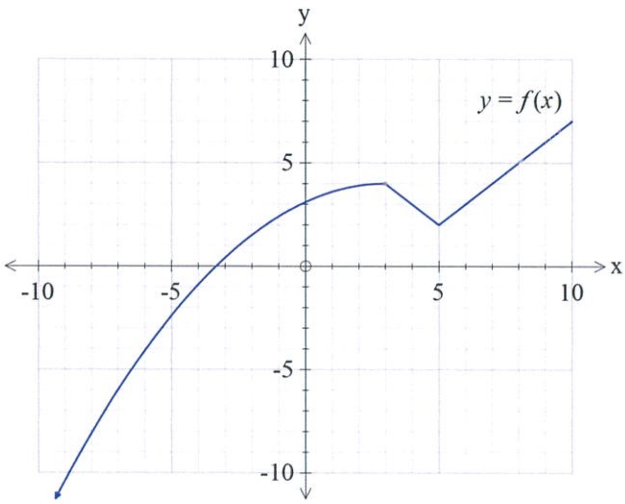
c) On the sketch above show the precise points where $f(x)=f^{-1}(x)$

✓ is on line $y=x$

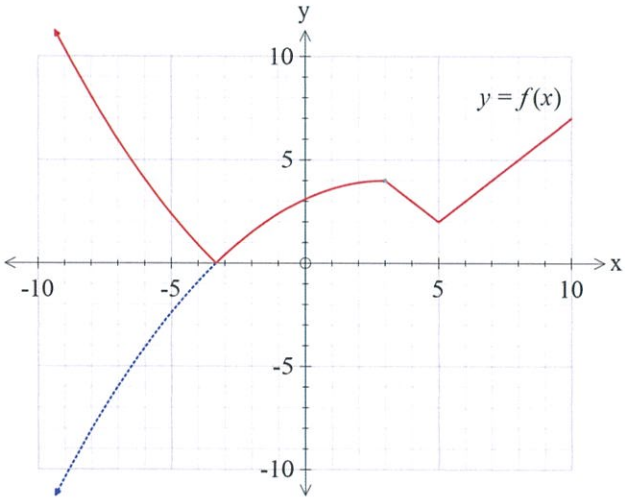
✓ $x \approx 2.2$ (± 0.3)

Q4 ~~(2 & 3 = 5 marks)~~

Q4 (2 & 3 = 5 marks)
Consider the function $y = f(x)$ for the questions below.

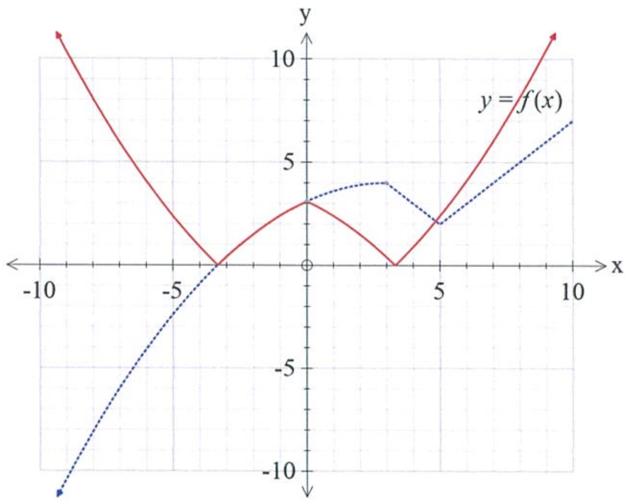


a) Sketch the function $y = |f(x)|$ on the axes below.



✓ unchanged for $f(x) > 0$
✓ reflected in x axis for $f(x) < 0$

b) Sketch the function $y = |f(-|x|)|$ on the axes below.



✓ left side of $f(x)$ reflected in y axis
✓ y intercept of 3
✓ negative parts reflected in x axis

Q5 (3 & 4 = 7 marks)

a) Two moving objects have the following position vectors and constant velocities at time, $t=0$:

$$r_a=\begin{pmatrix}9\\-8\end{pmatrix}m \quad v_a=\begin{pmatrix}-2\\7\end{pmatrix}m/s$$

$$r_b=\begin{pmatrix}11\\-3\end{pmatrix}m \quad v_b=\begin{pmatrix}5\\-3\end{pmatrix}m/s$$

$$\underset{\sim}{d}=\overrightarrow{AB}+t\,{}_B\vec{V}_A$$
$$=\begin{pmatrix}11\\-3\end{pmatrix}-\begin{pmatrix}9\\-8\end{pmatrix}+t\left[\begin{pmatrix}5\\-3\end{pmatrix}-\begin{pmatrix}-2\\7\end{pmatrix}\right]$$
$$=\begin{pmatrix}2\\5\end{pmatrix}+t\begin{pmatrix}7\\-10\end{pmatrix}$$

Determine the closest approach and the time that this will occur.

$$\underset{\sim}{d}\cdot{}_B\vec{V}_A=0$$
$$\begin{pmatrix}2+7t\\5-10t\end{pmatrix}\cdot\begin{pmatrix}7\\-10\end{pmatrix}=0$$
$$7(2+7t)-10(5-10t)=0$$
$$14+49t-50+100t=0$$
$$149t=36$$
$$t=\frac{36}{149}\ (0.242)$$

$|\underset{\sim}{d}|\doteq 4.51$ metres

✓ sets up dot product or displacement function

✓ solves for time (dot = 0 or calculus min)

✓ determines magnitude of closest approach.

b) Let the circle S have a radius 3 units and centre $(1,\beta)$, where β is a constant, and the line $r=\begin{pmatrix}-2\\0\end{pmatrix}+\lambda\begin{pmatrix}3\\-5\end{pmatrix}$ is tangential to this circle. Determine the value of β ~~and the vector equation of the circle~~ S.

$$\left|\underset{\sim}{r}-\begin{pmatrix}1\\ \beta\end{pmatrix}\right|=3$$
$$\left|\begin{pmatrix}-2+3\lambda\\0-5\lambda\end{pmatrix}-\begin{pmatrix}1\\ \beta\end{pmatrix}\right|=3$$
$$\left|\begin{matrix}-3+3\lambda\\-5\lambda-\beta\end{matrix}\right|=3$$

$$(-3+3\lambda)^2+(5\lambda+\beta)^2=9$$
$$9\lambda^2-18\lambda+9+25\lambda^2+10\beta\lambda+\beta^2-9=0$$
$$34\lambda^2+(10\beta-18)\lambda+\beta^2=0$$
$$(10\beta-18)^2-4(34)\beta^2=0$$
$$\beta=-5\pm\sqrt{34}\ (-10.83,\ 0.83)$$

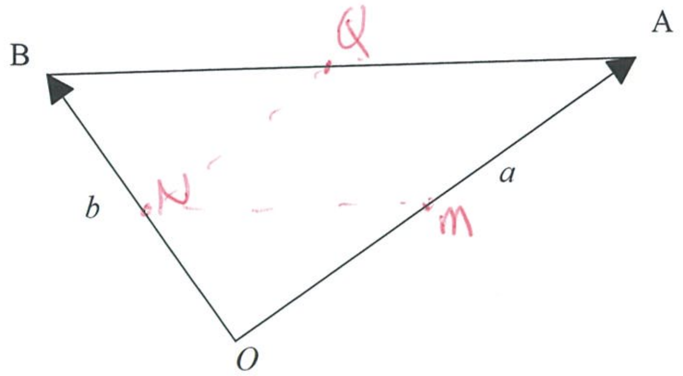
✓ sets up a vector eqn with λ and β

✓ sets up a quadratic eqn with λ and β

✓ uses zero determinant to solve for β

✓ States both values of β as an approx.

Q6 (1, 1, 1, 3, 1 & 3 =10 marks)
The diagram below shows a triangle with vertices with $O, A \& B$. Let O be the origin, with vectors $\overrightarrow{OA} = a$ and $\overrightarrow{OB} = b$.



a) Determine the following vectors in terms of $a \& b$.

i) $\overrightarrow{MA}$, where M is the midpoint of the line segment OA.

$\frac{1}{2}\underset{\frown}{a}$ ✓

ii) $\overrightarrow{BA}$ $= \underset{\frown}{a} - \underset{\frown}{b}$ ✓

iii) $\overrightarrow{AQ}$, where Q is the midpoint of the line segment AB. $\frac{1}{2}\overrightarrow{AB} = \frac{1}{2}(\underset{\frown}{b} - \underset{\frown}{a})$ ✓

Let N be the midpoint of the line segment OB.

b) Use a vector method tom prove that the quadrilateral $MNQA$ is a parallelogram.

$\overrightarrow{NM} = \overrightarrow{QA}$

$LHS = \overrightarrow{NM} = -\frac{1}{2}\underset{\frown}{b} + \frac{1}{2}\underset{\frown}{a}$

$\overrightarrow{QA} = \frac{1}{2}(\overrightarrow{BA}) = \frac{1}{2}(\underset{\frown}{a} - \underset{\frown}{b})$

$= -\frac{1}{2}\underset{\frown}{b} + \frac{1}{2}\underset{\frown}{a}$

$LHS = RHS$ ∴ Quadrilateral.

✓ states that opposite sides must be congruent + parallel (May use Vector statement)

✓ ~~does~~ obtains vector expressions for one pair of opposite sides

✓ shows that vectors are equal hence parallelogram

Q6 continued

Now consider the particular triangle OAB with $\overrightarrow{OA} = \begin{pmatrix} 3 \\ 2 \\ \sqrt{3} \end{pmatrix}$ and $OB = \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix}$ where α is a positive constant, chosen so that triangle OAB is isosceles, with $|\overrightarrow{OB}| = |\overrightarrow{OA}|$.

c) Show that $\alpha = 4$.

$$|\overrightarrow{OA}| = \sqrt{3^2 + 2^2 + 3}$$
$$= 4$$
$$= |\overrightarrow{OB}|$$
$$= \alpha$$

✓

d) Use a vector method to show that $\overrightarrow{OQ}$ is perpendicular to $\overrightarrow{AB}$.

$$\overrightarrow{OQ} = \underset{\sim}{b} + \tfrac{1}{2}\overrightarrow{BA}$$
$$= \underset{\sim}{b} + \tfrac{1}{2}(\underset{\sim}{a} - \underset{\sim}{b})$$
$$= \tfrac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b})$$

$$\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$$

$$\overrightarrow{OQ} \cdot \overrightarrow{AB} = \tfrac{1}{2}(\underset{\sim}{b} + \underset{\sim}{a})(\underset{\sim}{b} - \underset{\sim}{a})$$
$$= \tfrac{1}{2}(|\underset{\sim}{b}|^2 - |\underset{\sim}{a}|^2)$$
$$= 0 \quad \text{as} \quad |\underset{\sim}{b}| = |\underset{\sim}{a}|$$

✓ obtains expression for $\overrightarrow{OQ}$ in terms of $\underset{\sim}{a}$ & $\underset{\sim}{b}$

✓ obtains expression of dot product $\overrightarrow{OQ} \cdot \overrightarrow{AB}$ in terms of $\underset{\sim}{a}$ & $\underset{\sim}{b}$

✓ shows that dot product equals zero due to $|\underset{\sim}{a}| = |\underset{\sim}{b}|$

Q7 (5 marks)

Let $w=1+qi$ where q is a real constant. Let $p(z)=z^3+bz^2+cz+d$, where $b,c\,\&\,d$ are real constants. If $p(z)=0$ for $z=w$ and all roots of $p(z)=0$ satisfy $|z^3|=8$, determine all possible values of $q,b,c\,\&\,d$.

$$\left(\sqrt{1+q^2}\right)^3=8=2^3 \qquad z=\pm 2$$

$$1+q^2=2^2$$
$$q^2=3$$
$$q=\pm\sqrt{3}$$

$$(z-(1+\sqrt{3}i))(z-(1-\sqrt{3}i))$$
$$=z^2-2z+4$$

$z=2$

$$(z-2)(z^2-2z+4)=z^3-4z^2+8z-8$$

$q=\pm\sqrt{3}$ $b=-4$ $c=8$ $d=-8$

$z=-2$

$$(z+2)(z^2-2z+4)=z^3+8$$

$q=\pm\sqrt{3}$ $b=0$ $c=0$ $d=8$

✓ determines $q^2=3$

✓ determines $q=\pm\sqrt{3}$

✓ expands $(z-1-\sqrt{3}i)(z-1+\sqrt{3}i)=z^2-2z+4$

✓ determines $b=-4$ $c=8$ $d=-8$ with real soln $z=2$

✓ determines $b=0=c$ $d=8$ with real soln $z=-2$